

Thermodynamic Bethe Ansatz for the subleading magnetic perturbation of the tricritical Ising model

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Abstract

We give further support to Smirnov's conjecture on the exact kink S-matrix for the massive Quantum Field Theory describing the integrable perturbation of the $c = 0.7$ minimal Conformal Field theory (known to describe the tri-critical Ising model) by the operator $\phi_{2,1}$. This operator has conformal dimensions $(\frac{7}{16}, \frac{7}{16})$ and is identified with the subleading magnetic operator of the tri-critical Ising model. In this paper we apply the Thermodynamic Bethe Ansatz (TBA) approach to the kink scattering theory by explicitly utilising its relationship with the solvable lattice hard hexagon model. Analytically examining the ultraviolet scaling limit we recover the expected central charge $c = 0.7$ of the tri-critical Ising model. We also compare numerical values for the ground state energy of the finite size system obtained from the TBA equations with the results obtained by the Truncated Conformal Space Approach and Conformal Perturbation Theory.

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1 Introduction

Since the work [1] of A. B. Zamolodchikov, it is known that certain perturbations of conformal field theories (CFT's) lead to completely integrable models of massive quantum field theory (QFT). In this paper we consider the relevant perturbation of the $c = 0.7$ minimal CFT [2] (known to describe the tri-critical Ising model (TIM) [3]) by the operator $\phi_{2,1}$. This operator has conformal dimensions $(\Delta, \bar{\Delta}) = (\frac{7}{16}, \frac{7}{16})$ and is identified with the subleading magnetic operator of the TIM. The action of the perturbed tri-critical Ising model (PTIM) can be written as

$$\mathcal{A}_{PTIM} = \mathcal{A}_{TIM} + g \int \phi_{2,1}(x) d^2x \quad (1.1)$$

where \mathcal{A}_{TIM} represents the action of the $c = 0.7$ minimal CFT and g is a coupling constant, of dimension $(\text{mass})^{\frac{9}{8}}$, which describes the strength of the perturbation. It was shown in [1] that the action (1.1) describes a completely integrable QFT in the sense that it possesses an infinite number of non-trivial local integrals of motion.

This model has been already studied by several authors [4–7, 9]. Their results (with some violations in chronological order) are summarised below. The truncated conformal space approach (TCSA) calculations by Lässig, Mussardo and Cardy [5] showed that the theory (1.1) has two degenerate vacuum states (labelled here by 0 and 1) which correspond to the minima of the asymmetric double well Landau-Ginsburg potential. It was suggested [5] that the particle spectrum consists of a triplet of fundamental kink states¹ $|K_{01}\rangle$, $|K_{10}\rangle$, and $|K_{00}\rangle$ of the same mass m , which interpolate between these vacuum states (e.g. the kink $|K_{01}\rangle$ interpolates between the vacuum states 0 and 1, etc.). The fourth possible kink state $|K_{11}\rangle$ is not present due to the absence of the Z_2 -symmetry between the vacuum states which is broken by the perturbation term in (1.1). This leads to a restriction on the allowable adjacent vacuum states, which (as noticed by Zamolodchikov in [7]) is equivalent to the restriction imposed on adjacent spin states in the solvable lattice hard hexagon model (HHM) [8]. Therefore, assuming the above particle spectrum is exact, one can construct the kink-kink S -matrix from the associated Boltzmann weights of the (critical) HHM, since they satisfy the same Yang-Baxter equation. The normalisation of the S -matrix is then determined [9] by imposing unitarity, crossing symmetry and bootstrap requirements. The resulting S -matrix is given explicitly in the next Section. Alternatively, (and,

¹ Note that the vacuum labels 0 and 1 of paper [5, 6] have been interchanged here for consistency with the usual convention of spin labelling in the HHM [15].

in fact, earlier) Smirnov [4] has conjectured S -matrices for all $\phi_{1,2}$ and $\phi_{2,1}$ perturbed minimal models expressing them in terms of certain (appropriately normalised) RSOS projections of the $A_2^{(2)}$ R-matrix of the Izergin-Korepin model [10]. In particular, for the special case of the PTIM we are interested in here his result leads exactly [6] to the same particle structure and S -matrix as discussed above. It must, however, be emphasised that both of these approaches are still based on conjectures and as such require further verification.

In this paper we apply the thermodynamic Bethe Ansatz (TBA) approach [11, 12] to the conjectured S -matrix of the PTIM and calculate the ground state energy $E(R)$ in the finite-size geometry with the spatial coordinate compactified on a circle of circumference R . The calculations utilise the above relationship between the PTIM and the lattice hard hexagon model allowing the use of known results from the analytic Bethe-Ansatz solution [13] of the latter. The ground state energy $E(R)$ is expressed in terms of the solution of the TBA integral equation of an apparently new type. We show that in the “ultraviolet” limit $R \rightarrow 0$ the ground state energy scales as

$$E(R) \sim -\frac{\pi c}{6R} \quad (1.2)$$

with $c = 0.7$, exactly as one expects from (1.1). We also solve the TBA equations and evaluate the ground state energy $E(R)$ numerically. The results are in a good agreement with those obtained by the TCSA [5, 6] and from the Conformal Perturbation Theory (CPT) [14].

2 The Bethe-Yang equations

As mentioned above the kink-kink S -matrix in the PTIM is expressed in terms of the Boltzmann weights of the critical HHM. The latter is an “interaction-round-a-face” model on the square lattice. With a suitable normalisation convenient for our purposes the Boltzmann weights of this model can be written in the form [15]

$$W\left(\begin{array}{cc|c} 0 & 0 & u \\ 0 & 0 & \end{array}\right) = \frac{\sin \mu \sin(2\mu + u)}{\sin 2\mu \sin(\mu - u)} \quad (2.1)$$

$$W\left(\begin{array}{cc|c} 0 & 0 & u \\ 1 & 0 & \end{array}\right) = W\left(\begin{array}{cc|c} 0 & 1 & u \\ 0 & 0 & \end{array}\right) = \left[\frac{\sin \mu}{\sin 2\mu}\right]^{\frac{1}{2}} \frac{\sin u}{\sin(\mu - u)} \quad (2.2)$$

$$W\left(\begin{array}{cc|c} 1 & 0 & u \\ 0 & 0 & \end{array}\right) = W\left(\begin{array}{cc|c} 0 & 0 & u \\ 0 & 1 & \end{array}\right) = 1 \quad (2.3)$$

$$W\left(\begin{array}{cc|c} 0 & 1 & u \\ 1 & 0 & \end{array}\right) = \frac{\sin \mu \sin(2\mu - u)}{\sin 2\mu \sin(\mu - u)} \quad (2.4)$$

$$W\left(\begin{array}{cc|c} 1 & 0 & u \\ 0 & 1 & \end{array}\right) = \frac{\sin(\mu + u)}{\sin(\mu - u)} \quad (2.5)$$

where u is the spectral parameter and $\mu = \pi/5$. Note that this model is also solvable for other values of μ such that $5\mu/\pi = \text{integer}$, where it corresponds to some non-unitary QFT (see Sect. 6 below). For the above normalisation of the Boltzmann weights the partition function per site of the model in the limit of the infinite lattice is given by [15]

$$\kappa(u, \mu) = \exp\left(\int_{-\infty}^{\infty} \frac{\sinh(\pi - \mu)x \sinh 2ux}{2x \sinh \pi x \cosh \mu x} dx\right), \quad 0 \leq \text{Re } u \leq \mu. \quad (2.6)$$

In particular for $\mu = \pi/5$ it reads

$$\kappa(u, \pi/5) = \frac{\sin(\pi/5 + u) \sin(2\pi/5 - u)}{\sin(\pi/5 - u) \sin(2\pi/5 + u)}. \quad (2.7)$$

The kink-kink S -matrix proposed by Smirnov [4] can then be written as²

$$S_{\alpha\beta}^{\gamma\delta}(\theta) = \left(\frac{\rho_\gamma \rho_\delta}{\rho_\alpha \rho_\beta}\right)^{-\frac{\theta}{2\pi i}} R(\theta) W\left(\begin{array}{cc|c} \alpha & \delta & \lambda\theta \\ \gamma & \beta & \end{array}\right), \quad \alpha, \beta, \gamma, \delta \in \{0, 1\} \quad (2.8)$$

where θ is the rapidity variable, $\rho_0 = 1$, $\rho_1 = 2 \cos \mu$, while the other parameters are

$$\mu = \frac{\pi}{5}, \quad \lambda = \frac{9i}{5}. \quad (2.9)$$

The normalisation factor in (2.8) is given by

$$R(\theta) = \kappa(\lambda\theta, \mu)^{-1} F_{CDD}(\theta) \quad (2.10)$$

where $F_{CDD}(\theta)$ denotes the CDD factor

$$F_{CDD}(\theta) = -F_{-\frac{1}{9}}(\theta) F_{\frac{2}{9}}(\theta), \quad F_\alpha(\theta) = \frac{\sinh \theta + i \sin \alpha \pi}{\sinh \theta - i \sin \alpha \pi}. \quad (2.11)$$

² We follow [6] for the spin labelling convention in the elements of the S -matrix, except for the aforementioned interchange of vacuum labels 0 and 1.

It was shown [4, 6, 9] that the S -matrix (2.8) satisfies all the unitarity, crossing symmetry and bootstrap requirements.

Consider the state of N kinks distributed along a large spatial circle of the length L so that their average mutual distances are much greater than the correlation length $\xi = 1/m$ of the system, where m is the kink mass. Since the kink scattering processes are of the factorised type, this state can be described in terms of the Bethe wave function [11], [16]

$$\Psi(\theta_1, \dots, \theta_N) = \sum_{\{\alpha\}} \psi^{\{\alpha\}} |K_{\alpha_1 \alpha_2}(\theta_1) K_{\alpha_2 \alpha_3}(\theta_2) \dots K_{\alpha_N \alpha_1}(\theta_N)\rangle \quad (2.12)$$

where $\psi^{\{\alpha\}}$ denotes the “colour” wave function depending on the set of interkink vacuum labels $\{\alpha\} = \{\alpha_1, \alpha_2, \dots, \alpha_N\}$, $\alpha_i = 0, 1$, along the circle. Note that the absence of the kink K_{11} in the theory implies that the sum in (2.12) is taken over only those sequences of $\{\alpha\}$ which do not contain two consecutive α ’s equal to 1 in any place.

Consistency of the imposed periodic boundary conditions requires that

$$e^{imL \sinh \theta_k} \sum_{\{\alpha\}} T(\theta_k; \theta_1, \dots, \theta_N)_{\{\alpha\}}^{\{\beta\}} \psi^{\{\alpha\}} = -\psi^{\{\beta\}}, \quad k = 1, \dots, N \quad (2.13)$$

where

$$T(\theta; \theta_1, \dots, \theta_N)_{\{\alpha\}}^{\{\beta\}} = \prod_{i=1}^N S_{\beta_i \alpha_{i+1}}^{\alpha_i \beta_{i+1}}(\theta - \theta_i) \quad (2.14)$$

can be interpreted as the transfer matrix of a two-dimensional lattice model acting in the space of colour wave functions $\psi^{\{\alpha\}}$. The equations (2.13) result in constraints on the set of the kink rapidities $\theta_1, \theta_2, \dots, \theta_N$ which are known as the Bethe-Yang (BY) equations

$$e^{imL \sinh \theta_k} \Lambda(\theta_k; \theta_1, \dots, \theta_N) = -1, \quad k = 1, \dots, N \quad (2.15)$$

where $\Lambda(\theta; \theta_1, \dots, \theta_N)$ are the eigenvalues of the transfer matrix (2.14). From the definition (2.8) it is obvious that (2.14) is just the transfer matrix of the (inhomogeneous) critical HHM. The eigenvalues of the latter were found in [13] by solving the transfer matrix functional equation. Using this result one obtains

$$\Lambda(\theta; \theta_1, \dots, \theta_N) = \Lambda_{HHM}(\lambda\theta; \theta_1, \dots, \theta_N) \prod_{j=1}^N R(\theta - \theta_j) \quad (2.16)$$

where

$$\Lambda_{HHM}(u; \theta_1, \dots, \theta_N) = \omega \frac{Q(u + \mu)}{Q(u)} + \omega^{-1} f(u) \frac{Q(u - \mu)}{Q(u)}$$

$$f(u) = \prod_{j=1}^N \frac{\sin(u - \lambda\theta_j)}{\sin(u - \lambda\theta_j - \mu)}, \quad Q(u) = \prod_{k=1}^{N/2} \sin\left(u - \frac{\mu}{2} - \lambda\alpha_k\right) \quad (2.17)$$

$$\omega^5 = -(-1)^{N/2}, \quad \omega \neq \pm 1$$

and where N is assumed to be even, and the trivial difference in the normalisation of the transfer matrix (2.14) with that of ref. [13] has been taken into account. The (complex) numbers α_k , $k = 1, \dots, N/2$, are determined by the following Bethe-Ansatz equations (BAE)

$$\omega^{-2} \prod_{j=1}^N S_1(\alpha_k - \theta_j) = - \prod_{l=1}^n S_2(\alpha_k - \alpha_l) \quad (2.18)$$

where

$$n = \frac{N}{2} \quad (2.19)$$

and

$$S_j(\theta) = \frac{\sin(\lambda\theta + \mu j/2)}{\sin(\lambda\theta - \mu j/2)}. \quad (2.20)$$

Note that the equations (2.18) can be considered as the BAE for the $n = N/2$ up-arrow sector of the inhomogeneous six-vertex model [17] with “twisted” boundary conditions. The factor ω^{-2} in the RHS is then interpreted as the “twisting factor” determined by the value of the (horizontal) field applied to the six-vertex model.

3 The Thermodynamic Limit

According to the standard approach of the Thermodynamic Bethe Ansatz (e.g. [12], [18]) the BY equations (2.15) allow one to determine the spectral density of states of the infinite size system and hence to compute the free energy $f_0(mR)$ of this system at a finite temperature $1/R$. This free energy is then reinterpreted as the Casimir part of the ground state energy $E(R)$ of the finite-size system on a circle of the circumference R

$$E(R) - (\text{bulk term}) = Rf_0(mR) \quad (3.1)$$

where $f_0(mR)$ is the infinite-size free energy per unit length and the linear in R bulk energy term is known explicitly (see eq.(4.5), below).

In the thermodynamic limit the number of kinks N and the system size L simultaneously approach infinity. From earlier studies of the six vertex model it is well known [19] that for

$N \rightarrow \infty$ the roots α_k which solve the BAE (2.18) approach certain asymptotic patterns in the complex plane which can be viewed as collections of “strings”. Each string is a set of roots with the same real part, symmetric with respect to the real axis of α and equally spaced along the imaginary axis. The number of roots in a string is called the “length” of the string. In general the string spectrum of (2.18) may be very complicated [19] since it heavily depends on the arithmetic properties of the constant μ . For the case under consideration the constant μ is given by the simple fraction $\mu = \pi/5$ and the string spectrum of (2.18) consists of only five different strings. These are the strings of lengths $\ell = 1, 2, 3, 4$ (to be referred as 1-string, 2-strings, etc.) which are centred at the real axis $\Im m \alpha = 0$ and the “shifted 1-string” which is located on the line $\Im m \alpha = \pi/(2\lambda) \pmod{\pi/\lambda}$ and shifted from the real axis by half of the period of the function $Q(\lambda\theta)$. With these strings the roots α_k for a typical solution of (2.18) are given by

$$\alpha_{j,m}^{(\ell)} = \beta_j^{(\ell)} + \frac{\mu}{2\lambda}(\ell + 1 - 2m) + \delta(N) \pmod{\pi/\lambda}, \quad m = 1, \dots, \ell, \quad j = 1, \dots, n_\ell \quad (3.2)$$

and

$$\alpha_j^{(-)} = \beta_j^{(-)} + \frac{\pi}{2\lambda} \pmod{\pi/\lambda}, \quad j = 1, \dots, n_- \quad (3.3)$$

where $\beta_j^{(\ell)}$ and $\beta_j^{(-)}$ are (real) centres of the strings while n_ℓ , $\ell = 1, 2, 3, 4$ and n_- are the total numbers of strings of these types. The correction term $\delta(N)$ in (3.2) describes deviations of the roots from exact string positions. For large N it vanishes [20]³ as $O(\log N/N)$. In the case of the HHM we only require those solutions of (2.18) where the total number of roots is equal to $n = N/2$ and hence

$$\sum_{\ell=1}^4 \ell n_\ell + n_- = \frac{N}{2}. \quad (3.4)$$

For a generic value of the twist factor ω on the unit circle, $|\omega| = 1$, all the above strings are thermodynamically significant in the sense that for typical solutions of (2.18) with $n = N/2$ the ratios n_ℓ/N , $\ell = 1, \dots, 4$ and n_-/N remain finite as $N \rightarrow \infty$. However, when ω approaches the specific values $\log \omega = \pm i k \pi/5$, $k = 1, 2, 3, 4$ as it is required in (2.17) and (2.18) the situation drastically changes. It turns out in this case that, for many of these solutions, a finite number of strings moves away to infinity and the condition (3.4) can no longer be satisfied. More precisely, on the basis of numerical calculations it was conjectured [13] that the solutions

³Actually, the behaviour of $\delta(N)$ for large N is more complicated [20]. The asymptotics $\delta(N) \simeq O(\log N/N)$ quoted above are valid for the region $\text{Re}|\alpha| \ll \log N$ used for the thermodynamic limit below.

of (2.18), where the fractions of the 4-strings n_4/N and/or the shifted 1-strings n_-/N are not vanishing as $N \rightarrow \infty$, do not match the condition (3.4) and therefore have no relevance to the HHM. This leaves only three thermodynamically significant strings of lengths $\ell = 1, 2, 3$. Their total numbers are restricted by the relation

$$\sum_{\ell=1}^3 \ell n_\ell = \frac{N}{2} + o(N). \quad (3.5)$$

where the $o(N)$ term grows slower than N for $N \rightarrow \infty$. Assuming that the roots α_k take their limiting string form (3.2) one can rewrite the equation (2.15) as

$$\omega e^{imL \sinh \theta_k} \prod_{j=1}^N R(\theta_k - \theta_j) \prod_{\ell=1}^3 \prod_{j=1}^{n_\ell} S_\ell(\theta_k - \beta_j^{(\ell)}) = -1 + o(1), \quad (3.6)$$

where S_ℓ is defined in (2.20) and the correction term $o(1)$ vanishes as $N \rightarrow \infty$. The last equation looks very similar to the BY equation for the diagonal scattering theory. In fact, it is very convenient to formally interpret the string centres $\beta_j^{(\ell)}$ appearing in (3.6) as rapidities of some quasi-particles. These quasi-particles do not correspond to any observable particles in the asymptotic scattering states but rather reflect the nontrivial spin structure of the kink scattering. In the thermodynamic limit the rapidities of kinks and quasi-particles form dense distributions which can be described in terms of continuous densities. Let $\rho_0(\theta)$ denote the rapidity density of kink states and $\rho_\ell(\beta)$ denote such densities for the quasi-particles. We normalise these densities by the conditions

$$L \int_{-\infty}^{\infty} \rho_0(\theta) d\theta = N; \quad L \int_{-\infty}^{\infty} \rho_\ell(\beta) d\beta = n_\ell, \quad \ell = 1, 2, 3. \quad (3.7)$$

Following the standard procedure of the TBA one can now rewrite (3.6) as an integral equation

$$\frac{m}{2\pi} \cosh \theta = \rho_0(\theta) + \tilde{\rho}_0(\theta) - \sum_{k=0}^3 \int_{-\infty}^{\infty} \Psi_{0k}(\theta - \theta') \rho_k(\theta') d\theta' \quad (3.8)$$

where, as usual, $\tilde{\rho}_0(\theta)$ denotes the density of the “holes” [11] in the kink rapidity distribution and

$$\Psi_{00}(\theta) = \frac{1}{2\pi i} \partial_\theta \log R(\theta), \quad \Psi_{0j}(\theta) = \Psi_{j0}(\theta) = \frac{1}{2\pi i} \partial_\theta \log S_j(\theta), \quad j = 1, 2, 3. \quad (3.9)$$

Similarly the BA equations (2.18) lead to,

$$0 = \rho_j(\theta) + \tilde{\rho}_j(\theta) - \sum_{k=0}^3 \int_{-\infty}^{\infty} \Psi_{jk}(\theta - \theta') \rho_k(\theta') d\theta' \quad (3.10)$$

where $\tilde{\rho}_j(\theta)$, $j = 1, 2, 3$, denotes the densities of holes in the quasiparticle rapidity distributions, $\Psi_{j0}(\theta)$ is given above in (3.9) and

$$\Psi_{jk}(\theta) = -\frac{1}{2\pi i} \partial_\theta \sum_{m=1}^k \log [S_{j-k+2m}(\theta) S_{j+k-2m}(\theta)], \quad j, k = 1, 2, 3. \quad (3.11)$$

The equations (3.8) and (3.10) can be notably simplified. In fact, integrating the equation (3.10) with $j = 3$ over θ from $-\infty$ to ∞ and using (3.7) and the completeness relation (3.5) one obtains

$$\int_{-\infty}^{\infty} \tilde{\rho}_3(\theta) d\theta = 0. \quad (3.12)$$

This means that the corresponding density of holes identically vanishes, $\tilde{\rho}_3(\theta) \equiv 0$. Therefore the equation (3.10) with $j = 3$ can be used to exclude the density $\rho_3(\theta)$ from (3.8) and (3.10) expressing them through the remaining densities $\rho_j(\theta)$, $j = 0, 1, 2$. Another transformation, especially useful for a simplification of the integral kernel in these equations, consists in the following. The densities $\rho_j(\theta)$ and $\tilde{\rho}_j(\theta)$ enter the integral equation (3.10) in a non-symmetric way. Indeed, the hole densities $\tilde{\rho}_j(\theta)$ enter only the free term in (3.10) while the “particle” densities $\rho_j(\theta)$ enter the free term and also the integral term in (3.10). This arrangement can be changed. In particular, multiplying the equations (3.8) and (3.10) with a suitable matrix integral operator one can bring them to an equivalent form where the “particle” densities $\rho_j(\theta)$, for $j = 1, 2$ appear only in the free terms (the corresponding hole densities will then enter the free and the integral terms). Performing all the transformations described above and redenoting the set of densities,

$$\sigma_0(\theta) = \rho_0(\theta), \quad \tilde{\sigma}_0(\theta) = \tilde{\rho}_0(\theta); \quad \sigma_j(\theta) = \tilde{\rho}_j(\theta), \quad \tilde{\sigma}_j(\theta) = \rho_j(\theta), \quad j = 1, 2,$$

one obtains

$$\frac{m}{2\pi} \delta_{j,0} \cosh \theta = \sigma_j(\theta) + \tilde{\sigma}_j(\theta) + s_j \sum_{k=0}^2 \int_{-\infty}^{\infty} \Phi_{j,k}(\theta - \theta') \sigma_k(\theta') d\theta', \quad j = 0, 1, 2. \quad (3.13)$$

The new kernel reads

$$\Phi_{j,k}(\theta) = -(\delta_{j,k+1} + \delta_{j,k-1})\phi_0(\theta) - s_0 \delta_{j,0} \delta_{k,0} \phi_1(\theta) \quad (3.14)$$

$$\phi_0(\theta) = \frac{|\lambda|}{2\mu} \frac{1}{\cosh(\pi|\lambda|\theta/\mu)}, \quad \phi_1(\theta) = \frac{1}{2\pi i} \partial_\theta \log F_{CDD}(\theta) \quad (3.15)$$

where $F_{CDD}(\theta)$ is given by (2.11) and the quantities s_j , $j = 0, 1, 2$ are sign factors

$$s_0 = -\frac{\Im m \lambda}{|\Im m \lambda|}, \quad s_1 = s_2 = 1. \quad (3.16)$$

Of course, in the case considered where λ is given by (2.9), the value of s_0 is fixed to be $s_0 = -1$. We have given the formula (3.16) just to illuminate the origin of these sign factors in (3.13).

Thus, we have shown that in the thermodynamic limit the scattering state of the system of kinks in the PTIM is described by the three pairs of the rapidity and the hole densities $\sigma_j(\theta)$, $\tilde{\sigma}_j(\theta)$, $j = 0, 1, 2$, which are constrained by the integral equation (3.13). In the next section we will use this result to compute the equilibrium free energy of the system at a finite temperature.

4 The ground state energy of the finite size system

In the thermodynamic limit, the free energy for the system of kinks considered in the previous section is given by the functional

$$\mathcal{F}[\sigma_j(\theta), \tilde{\sigma}_j(\theta)] = \mathcal{E}[\sigma_0(\theta)] - \frac{1}{R} \mathcal{S}[\sigma_j(\theta), \tilde{\sigma}_j(\theta)], \quad (4.1)$$

where $1/R$ acts as the temperature parameter

$$\mathcal{E} = mL \int_{-\infty}^{\infty} \cosh \theta \sigma_0(\theta) d\theta, \quad (4.2)$$

is the energy of the system and \mathcal{S} is the combinatorial entropy [21] for the given set of the “particle” and “hole” densities $\sigma_j(\theta), \tilde{\sigma}_j(\theta)$, $j = 0, 1, 2$.

Following the standard calculations the equilibrium free energy

$$f_0(mR) = \frac{\mathcal{F}}{L} = -\frac{m}{2\pi R} \int_{-\infty}^{\infty} \cosh \theta \log(1 + e^{-\varepsilon_0(\theta)}) d\theta \quad (4.3)$$

is obtained by minimising the free energy functional (4.1) with respect to the above set of densities. The pseudo-energies

$$\varepsilon_j(\theta) = \log(\tilde{\sigma}_j(\theta)/\sigma_j(\theta))$$

are determined by the TBA integral equations

$$\varepsilon_j(\theta) = \delta_{j,0} mR \cosh \theta + \sum_{k=0}^2 s_k \int_{-\infty}^{\infty} \Phi_{j,k}(\theta - \theta') \log(1 + e^{-\varepsilon_k(\theta')}) d\theta' \quad (4.4)$$

where the kernel $\Phi_{jk}(\theta)$ and the sign factors s_k are given by (3.14) and (3.16) respectively. Combining (4.3) with (3.1) and inserting the known bulk energy term, one obtains the ground state energy $E(R)$ of the finite size system defined on a circle of circumference R

$$E(R) = \epsilon R - \frac{m}{2\pi} \int_{-\infty}^{\infty} \cosh \theta \log(1 + e^{-\varepsilon_0(\theta)}) d\theta. \quad (4.5)$$

where [14]

$$\epsilon = -\frac{\sqrt{3}m^2}{24 \sin(\frac{\pi}{18})} \approx -0.41560346108 \dots m^2. \quad (4.6)$$

As usual, the leading asymptotics of $E(R)$ in the ultraviolet limit, $R \rightarrow 0$, can be calculated by using the well known “dilogarithm trick” [22]. In this limit the pseudo-energies $\varepsilon_j(\theta)$ tend to finite constants $\varepsilon_j(0)$ inside the region $-\ln\left(\frac{2}{mR}\right) \ll \theta \ll \ln\left(\frac{2}{mR}\right)$. These constants satisfy the set of algebraic equations obtained by setting $R = 0$ in (4.4). Bringing them into a symmetric form one gets

$$y_j^{(0)} = \prod_{k=0}^2 \left(1 + y_k^{(0)}\right)^{l_{jk}} \quad (4.7)$$

where we have defined $y_0^{(0)} = \exp(-\varepsilon_0(0))$ and $y_j^{(0)} = \exp(\varepsilon_j(0))$, $j = 1, 2$. The coefficients l_{jk} are given by

$$\|l_{jk}\|_{0 \leq j, k \leq 2} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}. \quad (4.8)$$

The equations (4.7) have the unique positive solution

$$y_0^{(0)} = \sqrt{2}, \quad y_1^{(0)} = 1, \quad y_2^{(0)} = \frac{1}{\sqrt{2}}. \quad (4.9)$$

Next, in the limit $|\theta| \rightarrow \infty$, the pseudo-energy $\varepsilon_0(\theta) \rightarrow \infty$ while the other two pseudo-energies $\varepsilon_1(\theta), \varepsilon_2(\theta)$ tend to finite constants. These constants satisfy the set of equations

$$y_1^{(\infty)} = \left(1 + (y_2^{(\infty)})^{-1}\right)^{-\frac{1}{2}}, \quad y_2^{(\infty)} = \left(1 + (y_1^{(\infty)})^{-1}\right)^{-\frac{1}{2}}, \quad (4.10)$$

where we have defined $y_j^{(\infty)} = \exp(\varepsilon_j(\infty))$. These equations have the unique positive solution

$$y_0^{(\infty)} = 0, \quad y_1^{(\infty)} = y_2^{(\infty)} = \frac{\sqrt{5} - 1}{2} \quad (4.11)$$

where for convenience we have defined $y_0^{(\infty)} = \exp(-\varepsilon_0(\infty)) = 0$.

Following the standard calculations (see *e.g.* [16]) one can deduce the leading $R \rightarrow 0$ asymptotics of the ground state energy

$$E(R) \sim -\frac{1}{\pi R} \sum_{j=0}^2 \left[\mathcal{L} \left(f(y_j^{(\infty)}) \right) - \mathcal{L} \left(f(y_j^{(0)}) \right) \right] \quad (4.12)$$

where

$$\mathcal{L}(x) = -\frac{1}{2} \int_0^x \left[\frac{\ln(t)}{1-t} + \frac{\ln(1-t)}{t} \right] dt \quad (4.13)$$

is the Rogers dilogarithm function and $f(x) = 1/(1+x)$. With the simple dilogarithm identities

$$\mathcal{L}(x) + \mathcal{L}(1-x) = \mathcal{L}(1), \quad \mathcal{L}(1) = \frac{\pi^2}{6}, \quad \mathcal{L}\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} \quad (4.14)$$

the equation (4.12) reduces to

$$E(R) \sim -\frac{7\pi}{60R} \quad (4.15)$$

exactly as one expects from (1.2) with the central charge $c = 0.7$ of the tri-critical Ising CFT. The following few terms of the short distance expansion of $E(R)$ given by (4.5) can easily be determined numerically. This is done in the next Section where we compare the TBA results with the those of the Conformal Perturbation Theory (CPT)⁴.

5 Numerical results

The TBA equations (4.4) can be solved numerically using a simple iterative procedure. The corresponding ground state energy $E(R)$ is then determined by numerically evaluating the integral in (4.3). Some resulting values of $E(R)$ in the region $10^{-5} < mR < 10$ are given in Table 1 where they are compared to the corresponding values obtained from results of the Truncated Conformal Space Approach (TCSA) [23].

The TCSA calculations for the PTIM were performed in [5, 6], however the required numerical data was omitted there. We are indebted to Prof. G. Mussardo who provided us with the computer program [24] which generates the TCSA data presented in Table 1. Note that to relate the TCSA data back to TBA results one has to use the exact relation [14] between the kink

⁴It should be noted that bulk term in (4.5) exactly compensates the next-to-leading, linear in R , term in the short distance expansion of the integral in (4.5). In fact, the main correction term to (4.15) is of order $O(R^{9/4})$, exactly as predicted by CPT.

mR	$E(R)/m$ (TBA)	(TCSA)	(CPT)
0.00001	-36651.91429205	-36651.914292	-36651.9142920555
0.000025	-14660.7657173015	-14660.765717	-14660.7657173015
0.00005	-7330.38285968217	-7330.382860	-7330.38285968217
0.000075	-4886.92190775210	-4886.921907	-4886.92190775210
0.0001	-3665.19143229428	-3665.191432	-3665.19143229428
0.00025	-1466.07658143979	-1466.076578	-1466.07658143979
0.0005	-733.038309061764	-733.038300	-733.038309061764
0.00075	-488.692229111026	-488.692214	-488.692229111026
0.001	-366.519198155448	-366.519177	-366.519198155448
0.0025	-146.607830808513	-146.607765	-146.607830808515
0.005	-73.3042415739432	-73.304084	-73.3042415739653
0.0075	-48.8699046289368	-48.869643	-48.8699046290281
0.01	-36.6528965534080	-36.652522	-36.6528965536577
0.025	-14.6638535325716	-14.662677	-14.6638535387397
0.05	-7.33772690834948	-7.334929	-7.33772697813054
0.075	-4.89911302398585	-4.894469	-4.89911331240653
0.1	-3.68265799877659	-3.676005	-3.68265878811873
0.25	-1.52096720347295	-1.500051	-1.52098667429440
0.5	-0.86341841907386	-0.813667	-0.86363725527810
0.75	-0.70459594978559	-0.621994	-0.70548946787533
1.0	-0.67473619457720	-0.556359	-0.67713759026010
2.5	-1.07537196387997	-0.702079	-1.12306270607538
5.0	-2.08009259839356	-1.182952	-2.39571841186687
10.0	-4.15604421536308	-1.960064	-5.56031580714459

Table 1: Some numerical values for the ground state energy $E(R)$ (given in units of mass m) computed from the TBA equations (4.4),(4.5), from the TCSA method [24], and from the first three terms of the CPT short distance expansion (5.2), for different values of the dimensionless variable mR .

mass m and the coupling constant g in (1.1)

$$m = \frac{2^{\frac{1}{3}}\pi^{4/3}\Gamma(5/9)}{\Gamma(2/3)\Gamma(8/9)} \left(\frac{2\Gamma(11/16)}{\Gamma(5/16)\Gamma(3/4)} \right)^{8/9} g^{8/9} \approx 4.9277912244 \dots g^{8/9}. \quad (5.1)$$

The difference between the TCSA and TBA results given in Table 1 (seen more clearly in Fig. 1) is attributed to both the inherent inaccuracy of the TCSA truncation procedure (used to approximate the Hilbert space of energy states) which increases with R , and also to the fact that the dimension $\Delta = \frac{7}{16}$ is very close to the critical value of $\Delta = \frac{1}{2}$, above which divergence

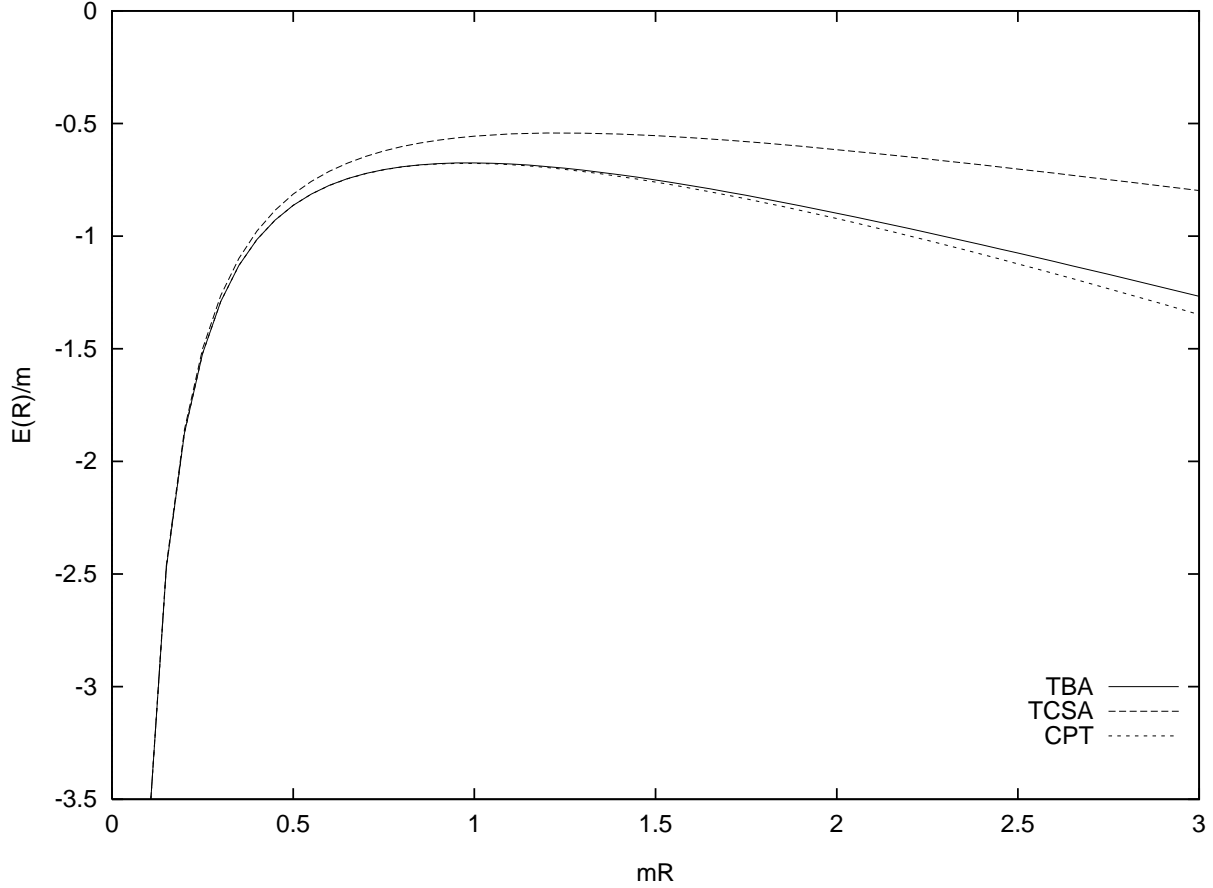


Figure 1: The ground state energy $E(R)$ (given in units of mass m) computed from the TBA equations (4.4),(4.5), from the TCSA method [24], and from the first three terms of the CPT short distance expansion (5.2), plotted against the dimensionless variable mR .

problems are known to occur. It is the latter problem which is thought to be responsible for the incorrect value of the bulk linear slope in the TCSA data at large R (as exemplified in Figure 1). We refer the reader to the original paper [5] where these problems are more thoroughly discussed.

In Table 1 (and Fig. 1) we also compare our numerical results for $E(R)$ with the first three terms of the short distance expansion

$$E(R) = -\frac{\pi}{6R} \left[\frac{7}{10} + \sum_{k=2}^{\infty} b_k (-\pi g)^k \left(\frac{R}{2\pi} \right)^{\frac{9k}{8}} \right]. \quad (5.2)$$

obtained from Conformal Perturbation Theory (CPT) [12, 16, 25]. In our case the coefficients b_2 and b_3 read explicitly [14]

$$b_2 = 12 \frac{\Gamma^2\left(\frac{7}{16}\right) \Gamma\left(\frac{1}{8}\right)}{\Gamma^2\left(\frac{9}{16}\right) \Gamma\left(\frac{7}{8}\right)} \approx 135.9255988883\dots, \quad b_3 = 0. \quad (5.3)$$

Considering these two coefficients as fixed and using the TBA results it is not difficult to make a numerical fit for a few following coefficients in (5.2). In particular, we found that

$$b_4 \approx -250.3 \dots, \quad b_5 \approx -0.58 \dots, \quad b_6 \approx 2.3 \dots \times 10^3. \quad (5.4)$$

6 Discussion

6.1 There are several other models of integrable QFT where the particle spectrum only consists of kink excitations with S -matrices related to the Boltzmann weights of the critical HHM. The most known among them is the RSOS(4) scattering theory arising from the perturbation of the ($c=0.8$) minimal CFT $\mathcal{M}_{5/6}$ by the operator $\phi_{1,3}$. The S -matrix of the general RSOS(p) scattering theory, $p = 3, 4, \dots$, just coincide with the “unitarised” Boltzmann weights⁵ of the critical lattice RSOS model of Andrews, Baxter and Forrester (ABF) [26]. This is an interaction-round-a-face model on the square lattice where admissible neighbouring spin states belong to adjacent sites of the A_p incidence diagram. The transfer matrix of this model commutes with the “spin reversal” operator generated by the reflection symmetry transformation of the incidence diagram and therefore it splits into a direct sum of two transfer matrices acting diagonally in the even and odd (with respect to the above symmetry) subspaces of the space of states. In the case $p = 4$ each of these two transfer matrices coincide [26] with the transfer matrix of the hard hexagon model (modulo minor differences in boundary conditions which are irrelevant in the thermodynamic limit). Therefore, for the purposes of the TBA calculations, the kink S -matrix of the RSOS(4) model can be replaced by an appropriate “hard hexagon” type S -matrix which is given by the formulae (2.7),(2.8) and (2.10) with λ and $F_{CDD}(\theta)$ replaced by

$$\mu = \frac{\pi}{5}, \quad \lambda = -\frac{i}{5}, \quad F_{CDD}(\theta) \equiv 1, \quad (6.1)$$

and all other quantities to remain unchanged. The calculation of Sect. 2.4 requires no modification for this case, one just needs to specialise the final results (4.4) and (4.5).

The ground state energy is

$$E(R) = (\text{bulk term}) - \frac{m}{2\pi} \int_{-\infty}^{\infty} \cosh \theta \log(1 + e^{-\varepsilon_0(\theta)}) d\theta. \quad (6.2)$$

⁵The Boltzmann weights should also be supplied with simple gauge transformation factors which ensure the crossing-symmetry of the S -matrix.

where the pseudo-energy $\varepsilon_0(\theta)$ is determined by the following TBA integral equations

$$\varepsilon_j(\theta) = \delta_{j,0} mR \cosh \theta + \sum_{k=0}^{p-2} \int_{-\infty}^{\infty} \Phi_{j,k}(\theta - \theta') \log(1 + e^{-\varepsilon_k(\theta')}) d\theta', \quad j = 0, 1, \dots, p-2. \quad (6.3)$$

with $p = 4$. The kernel $\Phi_{j,k}(\theta)$ is given by the same formula (3.14) which, with account of (6.1), reads explicitly

$$\Phi_{j,k}(\theta) = -\frac{1}{2\pi \cosh \theta} (\delta_{j,k+1} + \delta_{j,k-1}). \quad (6.4)$$

Note that the sign factors s_j which are present in (4.4) do not appear in (6.3) because the λ in (6.1) has the opposite sign to the one in (2.9).

The main technical point of our calculations was the diagonalization of the transfer matrix (2.14) by using the analytic Bethe-Ansatz solution [13] for the HHM. In fact, the aforementioned paper contains the results of this diagonalisation procedure for the complete hierarchy of ABF RSOS lattice models [26] (and their higher fused generalisations [27]). The calculations of the present paper are also easily extended to the case when the kink S -matrices are expressed through the Boltzmann weights of these more general models [28]. In particular, for the general RSOS(p) scattering theories one gets in this way the TBA equations (6.3), which are identical to those conjectured by Zamolodchikov [16]. Note also, that essentially the same set of TBA equations (or more precisely their lattice counterparts, which differ from (6.3) only in the form of the energy term $mR \cosh \theta$) arose previously [13] in the analysis of finite size corrections to the critical ABF RSOS models.

6.2 Another integrable QFT where the scattering theory is related to the HHM is obtained from the $c = -3/5$ minimal CFT $\mathcal{M}_{3/5}$ perturbed by the operator $\phi_{2,1}$ with dimension $\Delta = 3/4$. This QFT is also covered by Smirnov's conjecture [4]. It was specialised to this case in [29], resulting in a kink S -matrix that can be written exactly as in (2.10) and (2.12) with

$$\mu = \frac{3\pi}{5}, \quad \lambda = -\frac{3i}{5}, \quad F_{CDD}(\theta) = \frac{\sinh \theta + i \sin(\pi/3)}{\sinh \theta - i \sin(\pi/3)}. \quad (6.5)$$

Evaluating the integral in (2.6) for $\mu = 3\pi/5$ and substituting the result in the normalisation factor (2.10) one gets explicitly

$$R(\theta) = \frac{\sin\left(\frac{2\pi}{5} - \frac{3i\theta}{5}\right) \sin\left(\frac{\pi}{5} - \frac{3i\theta}{5}\right)}{\sin\left(\frac{2\pi}{5} + \frac{3i\theta}{5}\right) \sin\left(\frac{\pi}{5} + \frac{3i\theta}{5}\right)}. \quad (6.6)$$

The calculations of Sect. 3 and Sect. 4 can be repeated in this case with certain modifications. The main difference is related to the fact that for $\mu = 3\pi/5$ the string structure of the solutions

of the BAE equations (2.18) changes. Numerical calculations suggest that in this case only 1-strings in (3.2) and the shifted 1-strings (3.3) are thermodynamically significant. We believe this statement is correct and claim it as a conjecture. The rest of the calculations closely parallel those given in the case $\mu = \pi/5$ and requires no further assumptions. As a result one obtains that the ground state energy $E(R)$ is still given by the expression (6.2) while the pseudo-energy $\varepsilon_0(\theta)$ is determined by the following TBA equations

$$\begin{aligned}\varepsilon_0(\theta) &= mR \cosh \theta + \int_{-\infty}^{\infty} \Phi(\theta - \theta') \left(\log(1 + e^{-\varepsilon_0(\theta')}) - \log(1 + e^{-\varepsilon_1(\theta')}) \right) d\theta', \\ \varepsilon_1(\theta) &= \int_{-\infty}^{\infty} \Phi(\theta - \theta') \left(\log(1 + e^{-\varepsilon_1(\theta')}) - \log(1 + e^{-\varepsilon_0(\theta')}) \right) d\theta',\end{aligned}\quad (6.7)$$

where the kernel is given by

$$\Phi(\theta) = \frac{\sqrt{3} \sinh(2\theta)}{\pi \sinh(3\theta)} \quad (6.8)$$

Note also that

$$\Phi(\theta) = \frac{i}{2\pi} \partial_{\theta} \log F_{CDD}(\theta) \quad (6.9)$$

where F_{CDD} is the CDD factor defined in (6.5).

Apparently the same TBA equations for this QFT were conjectured in [30]⁶ from different arguments.

6.3 Yet another related QFT is obtained as the perturbation of the minimal CFT $\mathcal{M}_{5/6}$ by the operator $\phi_{1,2}$ of dimension $\Delta = \frac{1}{8}$. Again the S -matrix of the fundamental kink states (as given by Smirnov's conjecture [4]) is expressed through the Boltzmann weights of the HHM (2.3-2.7) with $\mu = \pi/5$. However, unlike all of the above examples, the bound state structure of this QFT appears to be extremely complicated. Apart from the fundamental kinks it contains higher kinks and breathers as bound states of the fundamental kinks. These higher kinks and breathers produce even more new kink and breather states and so on. A few of the lower kink and breather states were found in [34]. It would be interesting to see whether this bootstrap procedure closes

⁶In [30] the TBA equations for the $\mathcal{M}_{3/5}$ minimal CFT perturbed by the operator $\phi_{2,1}$ were obtained by “orbifolding” of the TBA equations associated with the product of $A_2 \otimes A_2$ Dynkin diagrams. The latter were previously conjectured in [31, 32] (in a rather general form) as field theory counterparts for the TBA equations of lattice RSOS models connected with the simply laced algebras obtained in ref. [33]. It should be noted that the relevant equations in refs. [30–32] contain some consistent misprints. However, tracing their derivations back to the original reference [33] one can recover the TBA equations (6.7) exactly as given here. We thank F. Ravanini and R. Tateo for explaining this issue.

at all. Particularly, some preliminary estimates show that the spectrum of this theory contains at least a dozen particles. In any case all of the kink-kink S -matrices are expressed in terms of the HHM Boltzmann weights (2.3-2.7) and therefore the TBA calculation of the present paper could be generalised to this case provided the bootstrap program is completed.

The above considerations were restricted only to the calculation of the ground state energy in the finite-size IQFT's by the TBA method. Recently, a few new approaches were introduced [35–39] which allow one to calculate also excited state energies. In ref. [37] it was shown that the functional equations for the IQFT “commuting transfer-matrices” introduced in [35] can be transformed [40] to the integral equations which generalise the equations of TBA to the excited states. The results of [37] apply to massive IQFT's obtained by $\phi_{1,3}$ perturbations of the minimal CFT's which are related to the $U_q(\widehat{sl}(2))$ quantum algebra. A suitable generalisation of this approach to treat the $\phi_{1,2}$ or $\phi_{2,1}$ perturbations of the minimal CFT's (in particular, the PTIM considered above) should be associated with the q -deformed twisted Kac-Moody algebra $A_2^{(2)}$. Some results in this direction were obtained in [41]. We hope to address this problem in future publications.

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References

- [1] A. B. Zamolodchikov: Integrals of Motion in Scaling 3-State Potts Model Field Theory, *Int. J. Mod. Phys.* **A3** (1988), 743-750; A. B. Zamolodchikov: Integrable Field Theory from Conformal Field Theory, *Advanced Studies in Pure Mathematics* **19** (1989), 641-674.
- [2] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov: Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory, *Nucl. Phys.* **B241** (1984), 333-380.

- [3] D. Friedan, Z. Qiu, and S. Shenker: Conformal Invariance, Unitarity, and Critical Exponents in Two Dimensions, *Phys. Rev. Lett.* **52** (1984), 1575-1578.
- [4] F. A. Smirnov: Exact S-Matrices for $\phi_{1,2}$ -Perturbed Minimal Models of Conformal Field Theory, *Int. J. Mod. Phys.* **A6** (1991), 1407-1428.
- [5] M. Lässig, G. Mussardo, and J. L. Cardy: The Scaling Region of the Tricritical Ising Model in Two Dimensions, *Nucl. Phys.* **B348** (1991), 591-618.
- [6] F. Colomo, A. Koubek, and G. Mussardo: The Sub-leading Magnetic Deformation of the Tricritical Ising Model in 2D as RSOS Restriction of the Izergin-Korepin Model, *Phys. Lett.* **B274** (1992), 367-373; F. Colomo, A. Koubek, and G. Mussardo: On the S Matrix of the Subleading Magnetic Deformation of the Tricritical Ising Model in Two Dimensions, *Int. J. Mod. Phys.* **A7** (1992), 5281-5305.
- [7] A. B. Zamolodchikov: S -matrix of the subleading magnetic perturbation of the tricritical Ising model, *Princeton Preprint* (1990), PUTP 1195-90.
- [8] R. J. Baxter: Hard Hexagons: Exact solution, *J. Phys.* **A13** (1980), L61-L70.
- [9] T. R. Klassen, and E. Melzer: Kinks in Finite Volume, *Nucl. Phys.* **B382** (1992), 441-485.
- [10] A. G. Izergin, and V. E. Korepin: The Inverse Scattering Method Approach to the Quantum Shabat-Mikhailov Model, *Commun. Math. Phys.* **79** (1981), 303-316.
- [11] C. N. Yang, and C. P. Yang: Thermodynamics of a One-Dimensional System of Bosons with Repulsive Delta-Function Interaction, *J. Math. Phys.* **10** (1969), 1115-1122.
- [12] A. B. Zamolodchikov: Thermodynamic Bethe Ansatz in Relativistic Models: Scaling 3-State Potts and Lee-Yang Models, *Nucl. Phys.* **B342** (1990), 695-720.
- [13] V. V. Bazhanov, and N. Yu. Reshetikhin: Critical RSOS Models and Conformal Field Theory, *Int. J. Mod. Phys.* **A4** (1989), 115-142.
- [14] V. A. Fateev: The Exact Relations between the Coupling Constants and the Masses of Particles for the Integrable Perturbed Conformal Field Theories, *Phys. Lett.* **B324** (1994), 45-51.

- [15] R. J. Baxter: The Inversion Relation Method for Some Two-Dimensional Exactly Solved Models in Lattice Statistics, *J. Stat. Phys.* **28** (1982), 1-41.
- [16] Al. B. Zamolodchikov: Thermodynamic Bethe Ansatz for RSOS Scattering Theories, *Nucl. Phys.* **B358** (1991), 497-523.
- [17] R. J. Baxter, *Exactly Solvable Models in Statistical Mechanics*, Academic Press, New York, 1982.
- [18] G. Mussardo: Off-critical Statistical Models: Factorised Scattering Theories and Bootstrap Program, *Phys. Rep.* **218** (1992), 215-379.
- [19] M. Takahashi, and M. Suzuki: One-Dimensional Anisotropic Heisenberg Model at Finite Temperature, *Prog. Theor. Phys.* **48** (1972), 2187-2209.
- [20] L. V. Avdeev, and B. D. Dörfel: Solutions of the Bethe Ansatz Equations for the XXX Antiferromagnet of Arbitrary Spin in the case of a Finite Number of Sites, *Theor. Math. Phys.* **71** (1987), 528-542.
- [21] C. N. Yang: Some Exact Results for the Many-Body Problem in One Dimension with Repulsive Delta-Function Interaction, *Phys. Rev. Lett.* **19** (1967), 1312-1315.
- [22] A. M. Tsvelick, and P. B. Wiegmann: Exact results in the theory of Magnetic Alloys, *Advances in Physics* **32** (1983), 453-713.
- [23] V. P. Yurov, and Al. B. Zamolodchikov: Truncated Conformal Space Approach to Scaling Lee-Yang Model, *Int. J. Mod. Phys.* **A5** (1990), 3221-3245; Truncated-Fermionic-Space Approach to the Critical 2D Ising Model with Magnetic Field, *Int. J. Mod. Phys.* **A6** (1991), 4557-4578.
- [24] M. Lässig, and G. Mussardo: Hilbert Space and Structure Constants of Descendant Fields in Two-Dimensional Conformal Theories, *Comp. Phys. Comm.* **66** (1991), 71-88.
- [25] T. Klassen, and E. Melzer: The Thermodynamics of Purely Elastic Scattering Theories and Conformal Perturbation Theory, *Nucl. Phys.* **B350** (1990), 635-689.

- [26] G. E. Andrews, R. J. Baxter, and P. J. Forrester: Eight-Vertex SOS Model and Generalized Rogers-Ramanujan-Type Identities, *J. Stat. Phys.* **35** (1984), 193-266.
- [27] E. Date, M. Jimbo, T. Miwa, and M. Okado: Fusion of the Eight Vertex SOS Model, *Lett. Math. Phys.* **12** (1986), 209-215.
- [28] T.J. Hollowood: From A_{m-1} trigonometric S -matrices to the thermodynamic Bethe Ansatz, *Phys. Lett.* **B320** (1994), 43-51.
- [29] G. Mussardo: Integrable Deformations of the Nonunitary Minimal Conformal Model $\mathcal{M}_{3,5}$, *Int. J. Mod. Phys.* **A7** (1992), 5027-5044.
- [30] F. Ravanini, M. Stanishkov, and R. Tateo: Integrable Perturbations of CFT with Complex Parameter: The $\mathcal{M}_{3/5}$ Model and its Generalizations, *Int. J. Mod. Phys.* **A11** (1996), 677-697.
- [31] F. Ravanini: Thermodynamic Bethe Ansatz for $\mathcal{G}_k \otimes \mathcal{G}_l / \mathcal{G}_{k+l}$ Coset Models Perturbed by their $\phi_{1,1,\text{Adj}}$ Operator, *Phys. Lett.* **B282** (1992), 73-79.
- [32] F. Ravanini, R. Tateo, and A. Valleriani: Dynkin TBA's, *Int. J. Mod. Phys.* **A8** (1993), 1707-1727.
- [33] V. V. Bazhanov, and N. Yu. Reshetikhin: Restricted Solid-on-Solid Models Connected with Simply Laced Algebras and Conformal Field Theory, *J. Phys.* **A 23** (1990), 1477-1492.
- [34] A. Koubek: S Matrices of $\phi_{1,2}$ -Perturbed Minimal Models: IRF Formulation and Bootstrap Program, *Int. J. Mod. Phys.* **A9** (1994), 1909-1927.
- [35] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov: Integrable Structure of Conformal Field Theory, Quantum KdV Theory and Thermodynamic Bethe Ansatz, *Commun. Math. Phys.* **177** (1996), 381-398.
- [36] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov: Integrable Structure of Conformal Field Theory II: Q-operator and DDV equation, *Preprint CLNS 96/1405, LPTENS 96/18, hep-th/9604044*.

- [37] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov: Integrable Quantum Field Theories in Finite Volume: Excited State Energies, *Nucl. Phys.* **B489** (1997), 487-531.
- [38] D. Fioravanti, A. Mariottini, E. Quattrini, and F. Ravanini: Excited State Destri-De Vega Equation for Sine-Gordon and Restricted Sine-Gordon Models, *Phys. Lett.* **B390** (1997), 243-251.
- [39] P. Dorey, and R. Tateo: Excited States by Analytic Continuation of TBA Equations, *Nucl. Phys.* **B482** (1996), 639-659.
- [40] A. Klümper, and P. Pearce: Conformal Weights of RSOS Lattice Models and Their Fusion Hierarchies, *Physica A* **183** (1992), 304-350.
- [41] D. Fioravanti, F. Ravanini, and M. Stanishkov: Generalized KdV and Quantum Inverse Scattering Description of Conformal Minimal Models, *Phys. Lett.* **B367** (1996), 113-120.